PERGAMON Computers and Mathematics with Applications 40 (2000) 81–93

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Moduli of Smoothness and Best Approximation of Functions with Singularities

M. GANZBURG
Department of Mathematics, Hampton University
Hampton, VA 23668, U.S.A.
michael.ganzburg@hamptonu.edu

Abstract—Error estimates for approximation of functions $\varphi_{\lambda,\alpha,0}(x) = \varphi_{\lambda,\alpha,1}(x) + i\varphi_{\lambda,\alpha,2}(x) = |x|^\lambda \exp(iA|x|^{-\alpha})$, $\lambda > 0$, $\alpha > 0$, $A \in \mathbf{R}$ are given. Let $E(f, B, L_p(\Omega))$ denote the error of approximation of f by elements from B in the L_p-metric. Then, it is shown that for polynomial approximation $E(\varphi_{\lambda,\alpha,i}, \mathcal{P}_n, L_p(-a,a)) \sim n^{-(\lambda+1/p)/(1+\alpha)}$ holds true for $1 \le p \le \infty$, $0 \le i \le 2$. The similar estimates are also valid for the errors of approximation by entire functions of exponential type, trigonometric polynomials, and periodic splines. The proofs are based on exact estimates of the moduli of smoothness of $\varphi_{\lambda,\alpha,i}$ and a general Stechkin-type theorem.

Keywords—Singularity, Best approximation, Moduli of smoothness, Polynomials.

1. INTRODUCTION

It was Vallée-Poussin who in 1910 began to study best approximation of functions with singularities. He obtained [1] some estimates of the error of polynomial approximation of $|x|$ which were improved by Bernstein [2].

Much attention has been paid to the study of the error of approximation of $|x|^\lambda$, by polynomials and entire functions of exponential type [3,4], by rational functions [5–7], by splines with variable knots [8], and by trigonometric polynomials with free spectrum [9].

Polynomial approximation of functions with "simple" singularities like $|x|^\lambda \log^k |x|$, $\sum_{i=1}^N |x - c_i|^\lambda$ and others were studied by Bernstein [10–12] and Nikolskii [13,14].

Brudnyi [15] developed a theory of rational and spline approximation of functions with singularities (see also [16]). In particular, he proved that the exact order of rational and spline approximation of $\varphi_{\lambda,\alpha,2}(x) = |x|^\lambda \sin(|x|^{-\alpha})$ in $L_p(-a,a)$ is $n^{-(\lambda+1/p)/\alpha}$.

Recently, we found [17] the exact order of approximation by polynomials and entire functions of exponential type of some infinitely differentiable functions like $\Phi_{\lambda,\alpha}(x) = |x|^\lambda \exp(iA|x|^{-\alpha})$, $\mathrm{Im}(A) > 0$.

In this paper, we study best approximation by algebraic and trigonometric polynomials, entire functions of exponential type and periodic splines of the following functions:

$$\varphi_{\lambda,\alpha,0}(x) = \varphi_{\lambda,\alpha,1}(x) + i\varphi_{\lambda,\alpha,2}(x) = |x|^\lambda \exp\left(iA|x|^{-\alpha}\right),$$
$$\psi_{\lambda,\alpha,i,s}(\tau) = \varphi_{\lambda,\alpha,i}(\sin\tau)\cos^s\tau, \qquad 0 \le i \le 2,$$

where $\lambda > 0$, $\alpha > 0$, $A \in R \setminus \{0\}$. We remark that $\varphi_{\lambda,0,0} = |x|^\lambda$ and $\varphi_{\lambda,\alpha,0} = \Phi_{\lambda,\alpha}$, $\mathrm{Im}(A) = 0$.

0898-1221/00/$ - see front matter Typeset by $\mathcal{AMS}$-TeX
PII: S0898-1221(00)00142-5

Main results on best approximation of $\varphi_{\lambda,\alpha,i}$ and $\psi_{\lambda,\alpha,i,s}$ are given in Section 3. In particular, we prove that the exact order of polynomial approximation of $\varphi_{\lambda,\alpha,i}$ in $L_p(-a,a)$ is $n^{-(\lambda+1/p)/(1+\alpha)}$. The proofs are based on estimates of moduli of smoothness of these functions (Section 2) and on a general Stechkin-type theorem (Section 3).

We use the following notation. Let $L_p(\Omega)$ be the Banach space of measurable functions f on a measurable set $\Omega \subseteq \mathbf{R}$ with norm $\|f\|_{L_p(\Omega)} = (\int_\Omega |f|\,dx)^{1/p}$ and let $L_\infty(\Omega)$ be the Banach space of continuous on $\Omega \subseteq \mathbf{R}$ functions f with norm $\|f\|_{L_\infty(\Omega)} = \sup_\Omega |f|$. Let $W_p^k(\Omega)$ be the set of all k-differentiable on $\Omega \subseteq \mathbf{R}$ functions f such that $f^{(k)} \in L_p(\Omega)$.

Throughout the paper, $C, C_1, C_2, \dots$ denote positive constants independent of $t, u, x, n, \sigma, \gamma, f, g$. The same symbol does not necessarily denote the same constant in different occurrences.

2. ESTIMATES OF MODULI OF SMOOTHNESS

Let Ω be one of the three sets, $\mathbf{R} = (-\infty, \infty)$, an interval $[a,b] \subset \mathbf{R}$, or the set $\mathbf{T}$ of real numbers modulo 2π. For $k \in \mathbf{N}$, we set

$$\Omega_h = \{x \in \Omega \mid x + kh \in \Omega\}, \qquad 0 < |h| \le \frac{(\operatorname{diam}\Omega)}{k}.$$

The set Ω_h coincides with Ω except when $\Omega = [a,b]$, in which case $\Omega_h = [a + k|h|, b]$ if $h < 0$, and $\Omega_h = [a, b - k|h|\,]$ if $h > 0$. Then, we can define for functions f on Ω,

$$\Delta_h^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh),$$

on the subset Ω_h. The k^{th} modulus of smoothness of a measurable function f is defined by

$$\omega_k(f,t,\Omega)_p = \omega_k(f,t)_p = \sup_{0<|h|\le t} \left\|\Delta_h^k f\right\|_{L_p(\Omega_h)}, \qquad 1 \le p \le \infty.$$

For $k \in \mathbf{N}$, $\lambda > 0$, $p \in [1,\infty]$, $t \ge 0$ we set

$$h_k(t,a) = \begin{cases} t^{(\lambda+1/p)/(1+\alpha)}, & \text{if } k > \dfrac{\lambda+1/p}{1+\alpha}, \\[2mm] t^k \ln^{1/p}\left(\dfrac{a}{t}\right), & \text{if } k = \dfrac{\lambda+1/p}{1+\alpha}, \\[2mm] t^k, & \text{if } k < \dfrac{\lambda+1/p}{1+\alpha}, \end{cases}$$

$$h_{k,0}(t,\infty) = h_{k,1}(t,\infty) = \begin{cases} t^{(\lambda+1/p)/(1+\alpha)}, & \text{if } k > \lambda + \dfrac{1}{p}, \quad 1 \le p \le \infty, \\[2mm] t^{(\lambda+1/p)/(1+\alpha)}, & \text{if } k = \lambda + \dfrac{1}{p}, \quad p = 1 \text{ or } p = \infty, \\[2mm] \infty, & \text{if } k < \lambda + \dfrac{1}{p}, \quad 1 \le p \le \infty, \\[2mm] \infty, & \text{if } k = \lambda + \dfrac{1}{p}, \quad 1 < p < \infty, \end{cases}$$

$$h_{k,2}(t,\infty) = \begin{cases} t^{(\lambda+1/p)/(1+\alpha)}, & \text{if } k > \lambda + \dfrac{1}{p} - \alpha, \quad 1 < p < \infty, \\[2mm] t^{\lambda/(1+\alpha)}, & \text{if } k \ge \lambda - \alpha, \quad p = \infty, \\ t^{(\lambda+1)/(1+\alpha)}, & \text{if } k \ge \lambda + 1 - \alpha, \quad p = 1, \quad k > 1 \text{ or } k = 1, \quad \lambda < \alpha, \\ t \ln t, & \text{if } k = \lambda + 1 - \alpha = 1, \quad p = 1, \\ \infty, & \text{if } k < \lambda + \dfrac{1}{p} - \alpha, \quad 1 \le p \le \infty, \\[2mm] \infty, & \text{if } k = \lambda + \dfrac{1}{p} - \alpha, \quad 1 < p < \infty. \end{cases}$$

In this section, we prove the following estimates for moduli of smoothness of $\varphi_{\lambda,\alpha,i}$ and $\psi_{\lambda,\alpha,i,s}$.

THEOREM 1. *Let $k \in \mathbf{N}$, $\lambda > 0$, $p \in [1,\infty]$, $s = 0,1,2,\dots$, $t \in (0, k^{-\alpha/(1+\alpha)})$ be given numbers.*

(a) *For $0 < a < \infty$, $0 \le i \le 2$,*

$$C_1 h_k(t,a) \le \omega_k(\varphi_{\lambda,\alpha,i}, t, [-a,a])_p \le C_2 h_k(t,a), \qquad t \in [0,a], \tag{2.1}$$

$$C_1 h_k(t,\pi) \le \omega_k(\psi_{\lambda,\alpha,i,s}, t, \mathbf{T})_p \le C_2 h_k(t,\pi), \qquad t \in [0,\pi]. \tag{2.2}$$

(b) *For $a = \infty$, $0 \le i \le 2$,*

$$C_1 h_{k,i}(t,\infty) \le \omega_k(\varphi_{\lambda,\alpha,i}, t, \mathbf{R})_p \le C_2 h_{k,i}(t,\infty), \qquad t \ge 0. \tag{2.3}$$

To prove Theorem 1, we need some auxiliary results. We begin by listing some elementary properties of the moduli of smoothness. Let f be a measurable function on $\Omega \subset \mathbf{R}$ such that $\omega_k(f,t)_p < \infty$ for all $t \ge 0$. Then,

(i) $\omega_k(f,t)_p$ is a nondecreasing function of t satisfying

$$\omega_k(f,\gamma t)_p \le (1+\gamma)^k \omega_k(f,t)_p; \tag{2.4}$$

(ii) for $t \ge 0$, the following triangle inequality holds:

$$\omega_k(f_1+f_2,t)_p \le \omega_k(f_1,t)_p + \omega_k(f_2,t)_p; \tag{2.5}$$

(iii) if $f \in L_p(\Omega)$, then

$$\omega_k(f,t)_p \le 2^k \|f\|_{L_p(\Omega)}; \tag{2.6}$$

(iv) if $f \in W_p^k(\Omega)$, then for $t > 0$,

$$\omega_k(f,t)_p \le t^k \left\| f^{(k)} \right\|_{L_p(\Omega)}; \tag{2.7}$$

(v) if f is not a polynomial of degree $k-1$, then

$$\omega_k(f,t)_p \ge C t^k, \qquad 0 < t < 1. \tag{2.8}$$

Concerning the proof of (2.4)–(2.8) we refer to [18, pp. 102–104; 19, pp. 44–46].

The following identities for $\varphi^{(m)}_{\lambda,\alpha,i}$ and $\psi^{(m)}_{\lambda,\alpha,i,s}$ may be verified easily by induction on m.

LEMMA 1.

(a) *For $\lambda > 0$, $\alpha > 0$, $1 \le i \le 2$, $x > 0$,*

$$\varphi^{(m)}_{\lambda,\alpha,0}(x) = \sum_{j=0}^{m} C_{j,m} \varphi_{\lambda-m-j\alpha,\alpha,0}(x), \tag{2.9}$$

$$\varphi^{(m)}_{\lambda,\alpha,i}(x) = \sum_{j=0}^{[m/2]} A_{j,m,i} \varphi_{\lambda-m-2j\alpha,\alpha,i}(x) + \sum_{j=1}^{[(m+1)/2]} B_{j,m,i} \varphi_{\lambda-m-(2j-1)\alpha,\alpha,3-i}(x), \tag{2.10}$$

(b) *for $\lambda > 0$, $\alpha > 0$, $1 \le i \le 2$, $\tau \in [0,\pi]$,*

$$\psi^{(m)}_{\lambda,\alpha,0,s}(\tau) = \sum_{j=0}^{m} \sum_{l=0}^{m-j} C_{l,j,m} \psi_{\lambda-m+2l-j\alpha,\alpha,0,s+m-2l}(\tau), \tag{2.11}$$

$$\begin{aligned} \psi^{(m)}_{\lambda,\alpha,i,s}(\tau) = {} & \sum_{j=0}^{[m/2]} \sum_{l=0}^{m-2j} A_{l,j,i,m} \psi_{\lambda-m+2l-2j\alpha,\alpha,i,s+m-2l}(\tau) \\ & + \sum_{j=0}^{[(m+1)/2]} \sum_{l=0}^{m-2j+1} B_{l,j,i,m} \psi_{\lambda-m+2l-(2j-1)\alpha,\alpha,3-i,s+m-2l}(\tau), \end{aligned} \tag{2.12}$$

where $C_{j,m}, C_{l,j,m}$ are complex constants and $A_{j,m,i}, B_{j,m,i}, A_{l,j,i,m}, B_{l,j,i,m}$ are real constants such that the coefficients with the maximal j in (2.9)–(2.12) are nonzero, and

$$\begin{aligned} C_{0,m} &= A_{0,m,1} = A_{0,m,2} = \lambda(\lambda-1)\dots(\lambda-k+1), \\ B_{1,m,2} &= -A\alpha\lambda(\lambda-1)\dots(\lambda-k+2). \end{aligned} \tag{2.13}$$

PROOF OF THEOREM 1. Upper estimates. The upper estimates in (2.1)–(2.3) are based on the decomposition of $\varphi_{\lambda,\alpha,i}$ and $\psi_{\lambda,\alpha,i,s}$ and inequalities for the corresponding K-functionals.

Let $\Omega = (-B,B)$ or $\Omega = [-B,B]$ be one of the three sets, $\mathbf{R}$, $[-a,a]$, or $\mathbf{T}$ and let $u \in (0,1)$ be a number such that $[-u,u] \subseteq \Omega$ and $u \geq kt$. For a measurable function f, we put $f_1(x) = \chi_{[-u,u]}(x)f(x)$, $f_2(x) = \chi_{\Omega\setminus[-u,u]}f(x)$, where χ_E denotes the characteristic function of E.

Assuming that $f_1 \in L_p(\Omega)$ and $\omega_k(f_2,t)_p < \infty$, we obtain from (2.5) and (2.6)

$$\begin{aligned} \omega_k(f,t,\Omega)_p &\leq \omega_k(f_1,t,\Omega)_p + \omega_k(f_2,t,\Omega)_p \\ &\leq 2^k\|f_1\|_{L_p(\Omega)} + \sup_{0<h\leq t}\left(\int_{-B}^{-u}|\Delta_h^k f(x)|^p\,dx + \int_{u-kh}^{B-kh}\left|\Delta_h^k f(x)\right|^p dx\right)^{1/p} \\ &\leq 2^k\left(\|f\|_{L_p(-u,u)} + \|f\|_{L_p(-2u,-u)} + \|f\|_{L_p(0,u)}\right) + \omega_k(f,t,(-B,-u))_p \\ &\quad + \omega_k(f,t,(u,B))_p. \end{aligned} \tag{2.14}$$

Next, assuming that f is an even function and $f \in W_p^k(u,B)$, we obtain from (2.14) and (2.7)

$$\omega_k(f,t,\Omega)_p \leq C\left(\|f\|_{L_p(0,2u)} + t^k\left\|f^{(k)}\right\|_{L_p(u,B)}\right). \tag{2.15}$$

In particular, since (2.9) and (2.10) yield the inequality

$$\left|\varphi_{\lambda,\alpha,i}^{(m)}(x)\right| \leq C\sum_{j=0}^{m} x^{\lambda-m-j\alpha}, \qquad 0\leq i\leq 2, \quad 0\leq m\leq k, \quad x>0, \tag{2.16}$$

we have from (2.15),(2.16)

$$\begin{aligned} \omega_k(\varphi_{\lambda,\alpha,i},t,[-a,a])_p &\leq C\left(\|\varphi_{\lambda,\alpha,i}\|_{L_p(0,2u)} + t^k\left\|\varphi_{\lambda,\alpha,i}^{(k)}\right\|_{L_p(u,a)}\right) \\ &\leq C\left(u^{\lambda+1/p} + t^k\left(h_k\left(u^{1+\alpha},a\right)u^{-k(1+\alpha)} + 1\right)\right). \end{aligned} \tag{2.17}$$

Next, choosing $u = t^{1/(1+\alpha)} > kt$ for $0 < t < k^{-\alpha/(1+\alpha)}$, we obtain from (2.17) $\omega_k(\varphi_{\lambda,\alpha,i},t,[-a,a])_p \leq Ch_k(t,a)$, giving the upper estimates in (2.1). Similarly (2.11), (2.12), and (2.15) yield the upper bounds in (2.2).

To prove the upper estimates in (2.3), we need to estimate $\|\varphi_{\lambda,\alpha,i}^{(k)}\|_{L_p(u,\infty)}$. It is easy to see that for $i = 0,1$ and $k > \lambda + 1/p$, $1 \leq p < \infty$ or $k \geq \lambda$, $p = \infty$, (2.16) yields

$$\left\|\varphi_{\lambda,\alpha,i}^{(k)}\right\|_{L_p(u,\infty)} \leq Cu^{\lambda-k(1+\alpha)+1/p} = Ch_{k,i}\left(u^{1+\alpha},\infty\right)u^{-k(1+\alpha)}. \tag{2.18}$$

Note that (2.18) is also valid for $i = 0,1$ and $k = \lambda+1$, $p = 1$, since by (2.13), $C_{0,\lambda+1} = A_{0,\lambda+1,1} = 0$ and, by (2.9) and (2.10),

$$\left|\varphi_{\lambda,\alpha,i}^{(m)}(x)\right| \leq C\sum_{j=1}^{m} x^{\lambda-m-j\alpha}, \qquad 0\leq i\leq 1, \quad x>0.$$

Then, (2.18) follows easily. Further, by (2.10),

$$\left|\varphi^{(k)}_{\lambda,\alpha,2}(x)\right| \le |H_k(x)| + C\sum_{j=2}^{k} x^{\lambda-k-j\alpha}, \qquad x > 0, \tag{2.19}$$

where

$$H_k(x) = A_{0,k,2}x^{\lambda-k}\sin(Ax^{-\alpha}) + B_{1,k,2}x^{\lambda-k-\alpha}\cos(Ax^{-\alpha}). \tag{2.20}$$

Taking account of the inequality $|\sin(Ax^{-\alpha})| \le |A|x^{-\alpha}$, we obtain from (2.19), (2.20), and (2.13)

$$\left|\varphi^{(k)}_{\lambda,\alpha,2}(x)\right| \le C\sum_{j=1}^{k} x^{\lambda-k-j\alpha}, \qquad x > 0. \tag{2.21}$$

Therefore, if $k > \lambda + 1/p - \alpha$, $1 \le p < \infty$, or $k \ge \lambda - \alpha$, $p = \infty$, then (2.21) yields

$$\left\|\varphi^{(k)}_{\lambda,\alpha,2}\right\|_{L_p(u,\infty)} \le Ch_{k,2}\left(u^{1+\alpha},\infty\right)u^{-k(1+\alpha)}. \tag{2.22}$$

It remains to prove (2.22) for $p = 1$, $k = \lambda + 1 - \alpha$. We have

$$\begin{aligned}|H_{\lambda+1-\alpha}(x)| &= |A|\alpha(\alpha+1)\dots(\alpha+k-1)x^{-1}\left|A^{-1}x^{\alpha}\sin\left(Ax^{-\alpha}\right) - \cos\left(Ax^{-\alpha}\right)\right| \\ &= Cx^{-1-2\alpha}\left(1 + O\left(x^{-4\alpha}\right)\right), \qquad x \to \infty.\end{aligned} \tag{2.23}$$

Then, (2.19) and (2.23) show that (2.22) is valid for $p = 1$, $k = \lambda + 1 - \alpha$. Next, combining (2.15), (2.18), and (2.22), we obtain

$$\omega_k(\varphi_{\lambda,\alpha,i}, t, (-\infty,\infty))_p \le C\left(u^{\lambda+1/p} + t^k h_{k,i}\left(u^{1+\alpha},\infty\right)u^{-k(1+\alpha)}\right). \tag{2.24}$$

Finally, choosing $u = t^{1/(1+\alpha)}$ in (2.24), we have

$$\omega_k(\varphi_{\lambda,\alpha,i}, t, (-\infty,\infty)) \le Ch_{k,i}(t,\infty), \qquad 0 \le i \le 2,$$

giving the upper bounds in (2.3).

LOWER ESTIMATES. For simplicity, we put $A = 1$. First, we prove the lower estimates in (2.1). For $1 \le k < (\lambda + 1/p)/(1+\alpha)$ they follow from (2.8). Let now $k \ge (\lambda + 1/p)/(1+\alpha)$ be an odd number and let $0 < t < t_0 = \min(1, aC_0)$, where $C_0 = (2^{(1+\alpha)/\alpha+2}\pi^{1/\alpha}\alpha k)^{-1} < 1$. Next, we consider the pairwise disjoint family of intervals $[a_s, b_s] \subseteq [0, a]$, where $a_s = a(2s\pi + \pi/4)^{-1/\alpha}$, $b_s = a(2s\pi)^{-1/\alpha}$, $s = 1, 2, \dots$.

Then there exists $s_0 \in \mathbf{N}$ such that

$$t_0(s_0+1)^{-(1+\alpha)/\alpha} \le t < t_0 s_0^{-(1+\alpha)/\alpha}. \tag{2.25}$$

Next, for $s = 1, 2, \dots$,

$$b_s - a_s = a\pi^{-1/\alpha}\left((2s)^{-1/\alpha} - \left(2s + \frac{1}{4}\right)^{-1/\alpha}\right) \ge Cs^{-(1+\alpha)/\alpha}, \tag{2.26}$$

$$a\left(\left(2s\pi - \frac{\pi}{4}\right)^{-1/\alpha} - (2s\pi)^{-1/\alpha}\right) \ge aC_0ks^{-(1+\alpha)/\alpha}. \tag{2.27}$$

Now, by (2.25) and (2.27), for all $s \in \mathbf{N}$, $1 \le s \le s_0$,

$$b_s + kt \le a(2s\pi)^{-1/\alpha} + t_0ks_0^{-(1+\alpha)/\alpha} \le a(2s\pi)^{-1/\alpha} + aC_0ks_0^{-(1+\alpha)/\alpha} \le a\left(2s\pi - \frac{\pi}{4}\right)^{-1/\alpha}. \tag{2.28}$$

Therefore,

$$\min_{x\in[a_s,b_s+kt]} \cos(x^{-\alpha}) \geq 2^{-1/2}. \tag{2.29}$$

Next, using (2.10), (2.25), (2.26), (2.28), and (2.29), we obtain

$$\begin{aligned}
\omega_k(\varphi_{\lambda,\alpha,2},t,[-a,a])_p &\geq \left\|\Delta_t^k\varphi_{\lambda,\alpha,2}\right\|_{L_p(0,a)} \geq \left(\sum_{s=1}^{s_0}\int_{a_s}^{b_s}\left|\Delta_t^k\varphi_{\lambda,\alpha,2}(x)\right|^p dx\right)^{1/p} \\
&= \left(\sum_{s=1}^{s_0}\int_{a_s}^{b_s}\left|\int_0^t\cdots\int_0^t \varphi_{\lambda,\alpha,2}^{(k)}(x+u_1+\cdots+u_k)\,du_1\ldots du_k\right|^p dx\right)^{1/p} \\
&\geq C\left(\sum_{s=1}^{s_0}\int_{a_s}^{b_s}\left|\int_0^t\cdots\int_0^t \varphi_{\lambda-k(1+\alpha),\alpha,1}(x+u_1+\cdots+u_k)\,du_1\ldots du_k\right|^p dx\right)^{1/p} \\
&\quad - C_1\sum_{j=0}^{k-1}\sum_{s=1}^{s_0}\left(\int_{a_s}^{b_s}\left(\int_0^t\cdots\int_0^t(|\varphi_{\lambda-k-js,\alpha,1}(x+u_1+\cdots+u_k)|\right.\right. \\
&\quad + \left.\left.|\varphi_{\lambda-k-js,\alpha,2}(x+u_1+\cdots+u_k)|)\,du_1\ldots du_k\right)^p dx\right)^{1/p} \\
&\geq C\left(\sum_{s=1}^{s_0}t^{kp}(b_s-a_s)(b_s+kt)^{(\lambda-k(1+\alpha))p}\right)^{1/p} - C_1\left(\sum_{j=0}^{k-1}\sum_{s=1}^{s_0}t^{kp}(b_s-a_s)a_s^{\lambda-k-j\alpha}\right)^{1/p} \\
&\geq Ct^k\left(\sum_{s=1}^{s_0}s^{-(1+\alpha)/\alpha-(\lambda-k(1+\alpha))p/\alpha}\right)^{1/p} - C_1t^k\left(\sum_{j=0}^{k-1}\sum_{s=1}^{s_0}s^{-(1+\alpha)/\alpha-(\lambda-k-j\alpha)p/\alpha}\right)^{1/p} \\
&\geq C\begin{cases} t^k s_0^{-(\lambda-k(1+\alpha)+1/p)/\alpha}\left(1-s_0^{-1}\ln^{1/p}s_0\right), & \text{if } k>\dfrac{\lambda+1/p}{1+\alpha} \\ t^k\left(\ln^{1/p}s_0\right)\left(1-\dfrac{1}{s_0}\right), & \text{if } k=\dfrac{\lambda+1/p}{1+\alpha}\end{cases} \geq Ch_k(t,a).
\end{aligned}$$

Thus, the lower estimate in (2.1) is established for $i=2$ and all odd $k\geq 1$. Similarly, we can obtain the lower bounds in (2.1) for $i=2$ and all even $k\geq 2$ and also for $i=1$, $k\in\mathbf{N}$. Using the same method, it is easy to derive the lower estimates in (2.2) for $i=1$ and $i=2$ from (2.8) and (2.12). The lower bounds in (2.1) and (2.2) for $i=0$ follows easily from the inequalities

$$\begin{aligned}
\omega_k(\varphi_{\lambda,\alpha,0},t,[-a,a])_p &\geq \omega_k(\varphi_{\lambda,\alpha,1},t,[-a,a])_p, \qquad 0<a\leq\infty, \\
\omega_k(\psi_{\lambda,\alpha,0,s},t,[-\pi,\pi])_p &\geq \omega_k(\psi_{\lambda,\alpha,1,s},t,[-\pi,\pi])_p.
\end{aligned} \tag{2.30}$$

Then, the inequalities

$$\omega_k(\varphi_{\lambda,\alpha,i},t,(-\infty,\infty))_p \geq \omega_k(\varphi_{\lambda,\alpha,i},t,[-a,a])_p, \tag{2.31}$$

and the lower estimates in (2.1) show that the lower bounds in (2.3) are valid for $i=0,2$ and $k>\lambda+1/p$, $1\leq p\leq\infty$, or $k=\lambda+1/p$, $p=1,\infty$, since $k\geq\lambda+1/p$ implies $k>(\lambda+1/p)/(1+\alpha)$, $1\leq p\leq\infty$. Next, we note that $k>\lambda+1/p-\alpha$ implies $k>(\lambda+1/p)/(1+\alpha)$, then $k=\lambda+1/p-\alpha$ and $k>1$ imply $k>(\lambda+1/p)/(1+\alpha)$, and finally $1=\lambda+1/p-\alpha$ implies $1=(\lambda+1/p)/(1+\alpha)$. Therefore, (2.31) and the lower estimates in (2.1) yield the lower bounds in (2.3) for $i=2$ and $k>\lambda+1/p-\alpha$, $1<p<\infty$ or $k\geq\lambda+1/p-\alpha$, $p=1,\infty$. It remains to consider the case when $\omega_k(\varphi_{\lambda,\alpha,i},t,(-\infty,\infty))_p=\infty$.

Suppose first $i=1$. Then, taking account of the inequality $\cos(x^{-\alpha})\geq 2^{-1/2}$ for $x\geq M=(4/\pi)^{1/\alpha}$ and using (2.10), we obtain for $k<\lambda+1/p$, $1\leq p\leq\infty$,

$$\omega_k(\varphi_{\lambda,\alpha,1},t,(-\infty,\infty))_p \geq \omega_k(\varphi_{\lambda,\alpha,1},t,(M,\infty))_p$$

$$
\begin{aligned}
&\geq \lim_{N\to\infty}\left(\int_M^N\left|\int_0^t\cdots\int_0^t \varphi_{\lambda,\alpha,1}^{(k)}(x+u_1+\cdots+u_k)\,du_1\ldots du_k\right|^p dx\right)^{1/p}\\
&\geq \lim_{N\to\infty}|A_{0,k,1}|\left(\int_M^N\left(\int_0^t\cdots\int_0^t \varphi_{\lambda,\alpha,1}^{(k)}(x+u_1+\cdots+u_k)^{\lambda-k}\right.\right.\\
&\quad\left.\times\ \cos\left((x+u_1+\cdots+u_k)^{-\alpha}\right)\,du_1\ldots,du_k\right)^p dx\Bigg)^{1/p}\\
&\quad - C\sum_{j=1}^k\left(\int_M^N\left(\int_0^t\cdots\int_0^t (x+u_1+\cdots+u_k)^{\lambda-k-j\alpha}\,du_1\ldots du_k\right)^p dx\right)^{1/p}\\
&\geq \lim_{N\to\infty}\left(2^{-1/2}\lambda(\lambda-1)\ldots(\lambda-k+1)t^k\left(\int_M^N x^{(\lambda-k)p}\,dx\right)^{1/p}\right.\\
&\quad\left.- C\sum_{j=1}^k\left(\int_M^N (x+kt)^{(\lambda-k-j\alpha)p}\,dx\right)^{1/p}\right)\\
&\geq C\lim_{N\to\infty}\left(N^{\lambda-k+1/p}-\sum_{j=1}C_jN^{\lambda-k-j\alpha+1/p}\right)=\infty.
\end{aligned}
$$

Similarly, if $k=\lambda+1/p$, $1<p<\infty$. Using (2.30) for $a=\infty$, we complete the proof of the lower estimates in (2.3) for $i=0$ and $i=1$.

Let now $i=2$ and let $k\leq\lambda+1/p-\alpha$, $1<p<\infty$ or $k<\lambda+1/p-\alpha$, $p=1,\infty$. Then, denoting $\delta=\alpha/(\lambda-k+1)$, we have $\delta\in(0,1)$. Next, taking into account the inequality $\sin(x^{-\alpha})\geq x^{-\alpha}$ which holds for all $x\geq M$ when M is large enough and using (2.10), we obtain

$$
\begin{aligned}
&\omega_k(\varphi_{\lambda,\alpha,2},t,(-\infty,\infty))_p\geq\omega_k(\varphi_{\lambda,\alpha,2},t,(M,\infty))_p\\
&\geq \lim_{N\to\infty}\left(\left(\int_M^N\left|\int_0^t H_k(x+u_1+\cdots+u_k)\,du_1\ldots du_k\right|^p dx\right)^{1/p}\right.\\
&\quad\left.- C\sum_{j=2}^k\left(\int_M^N\left(\int_0^t\cdots\int_0^t (x+u_1+\cdots u_k)^{\lambda-k-j\alpha}\,du_1\ldots du_k\right)^p dx\right)^{1/p}\right)\\
&\geq \lim_{N\to\infty}\left((\delta A_{0,k,2}-B_{1,k,2})\,t^k\left(\int_M^N x^{(\lambda-k-\alpha)p}\,dx\right)^{1/p}\right.\\
&\quad\left.- C\sum_{j=2}^k\left(\int_M^N (x+kt)^{(\lambda-k-j\alpha)p}\,dx\right)^{1/p}\right)\\
&\geq \lim_{N\to\infty}\left((C\lambda(\lambda-1)\ldots(\lambda-k+2)(\delta(\lambda-k+1)-\alpha)N^{\lambda-k-\alpha+1/p}\right.\\
&\quad\left.-\sum_{j=2}^k C_jN^{\lambda-k-j\alpha+1/p}\right)=\infty,
\end{aligned}
$$

where H_k, $A_{0,k,2}$, and $B_{1,k,2}$ are defined by (2.20) and (2.13). Thus, Theorem 1 is proved. ∎

REMARK 1. This method can be applied to many functions having isolated singularities such as $\operatorname{sgn} x\varphi_{\lambda,\alpha,i}(x)$, $\operatorname{sgn} x\psi_{\lambda,\alpha,i,s}(x)$, $|x|^\lambda\exp(i\ln^s|x|)$, $\cos(\exp(x))\cosh^{-\alpha}x$, and others.

REMARK 2. Functions $\varphi_{\lambda,\alpha,i}$ do not belong to $L_p(\mathbf{R})$ but (2.3) and (2.4) show that

$$\omega_k(\varphi_{\lambda,\alpha,i}, t, \mathbf{R})_p < \infty,$$

for all $t \geq 0$ and $k > \lambda + 1/p$.

3. ESTIMATES OF THE ERRORS OF APPROXIMATION

Let Ω be the three sets, $\mathbf{R}$, $[a,b]$, or $\mathbf{T}$, and S be a linear set of measurable functions on Ω. The error of approximation of a measurable function f on Ω by elements from S in the metric of $L_p(\Omega)$, $1 \leq p \leq \infty$, is

$$E(f, S, \Omega)_p = \inf_{g \in S} \|f - g\|_{L_p(\Omega)}.$$

Let M be an unbounded subset of $(0, \infty)$, $\{S_\gamma\}_{\gamma \in M}$ be an increasing family of linear sets of measurable functions on Ω, and let f be a measurable function on Ω. In what follows, we determine conditions of Jackson-Stechkin type which f and $\{S_\gamma\}_{\gamma \in M}$ satisfy, and starting from these conditions, we obtain a general Stechkin-type theorem for a lower estimate of the error of approximation of f. Then, we apply this theorem for obtaining the precise order of approximation of $\varphi_{\lambda,\alpha,i}$ and $\psi_{\lambda,\alpha,i,s}$ by trigonometric and algebraic polynomials, entire functions of exponential type, and splines.

(A) The Jackson-type inequality holds

$$E(f, S_\gamma, \Omega)_p \leq C\omega_k\left(f, \gamma^{-1}, \Omega\right)_p, \qquad \gamma \in M, \quad \gamma > 1. \tag{3.1}$$

(B) There exists a subset $\Omega_1 \subseteq \Omega$ such that for each $g \in S_\gamma$, $\gamma \in M$, with $\omega_k(g, t, \Omega)_p < \infty$, $t \in (0,1)$,

$$\omega_k(g, t, \Omega)_p \leq C(\gamma t)^k \omega_k\left(g, \gamma^{-1}, \Omega\right)_p. \tag{3.2}$$

It is easy to see that if S_γ consists of k-differentiable functions on Ω, then Condition (B) is a consequence of (2.7) and the following Stechkin-type condition.

(C) There exists a subset $\Omega_1 \subseteq \Omega$ such that for each $g \in S_\gamma$, $\gamma \in M$, with $\omega_k(g, t, \Omega)_p < \infty$, $t \in (0,1)$,

$$\left\|g^{(k)}\right\|_{L_p(\Omega_1)} \leq C\gamma^k \omega_k\left(g, \gamma^{-1}, \Omega\right)_p. \tag{3.3}$$

Next, let $h(t)$ be a continuous nondecreasing function on $(0, \infty)$, $h(0) = 0$, satisfying the H-condition, there exists $H > 1$ such that

$$\limsup_{t \to 0} \frac{h(2t)}{h(t)} = H. \tag{3.4}$$

We remark that (3.4) implies the following property of h, for any $\varepsilon > 0$ there exists t_0 such that

$$h(ut) \geq H^{-1} 2^{-\varepsilon} u^{\log_2 H + \varepsilon} h(t), \qquad 0 < t < t_0, \quad 0 < u \leq 1. \tag{3.5}$$

To prove (3.5), we note that for any $\varepsilon_1 > 0$, there exists t_0 such that for all $t \in (0, t_0)$ and all $s \in \mathbf{N}$,

$$\frac{h\left(2^{1-s} t\right)}{h\left(2^{-s} t\right)} < H + \varepsilon_1. \tag{3.6}$$

If $u \in (2^{-k}, 2^{-k+1}]$, then multiplying inequalities (3.6) for $1 \leq s \leq k$, we obtain

$$h(t) < (H + \varepsilon_1)^k h\left(2^{-k} t\right) \leq (H + \varepsilon_1)^{-1} u^{-\log_2(H + \varepsilon_1)} h(ut). \tag{3.7}$$

Then, (3.5) follows from (3.7) for $\varepsilon_1 = (2^\varepsilon - 1)H$.

Next, we remark that if $1 < H < 2^k$, then there exists $\mu \in (0, k)$ such that

$$h(ut) \ge Cu^\mu h(t), \qquad 0 < u \le 1, \quad 0 < t \le 1. \tag{3.8}$$

It is easy to show that (3.8) follows from (3.5) for $\varepsilon = (k - \log_2 H)/2$, $\mu = \log_2 H + \varepsilon$, and $0 < t \le t_0 \le 1$. If $t_0 < t \le 1$, then using (3.5) we have

$$h(ut) \ge h(ut_0) \ge C\left(\frac{ut_0}{t}\right)^\mu h(t) = Cu^\mu h(t),$$

giving (3.8).

THEOREM 2. *Let f be a measurable function on Ω and let Ω_1 be a subset of Ω. Let f and $\{S_\gamma\}_{\gamma\in M}$ satisfy Conditions (A) and (B) (or Conditions (A) and (C)) and let h satisfy the H-condition. If the inequalities*

$$C_1 h(t) \le \omega_k(f, t, \Omega_1)_p \le \omega_k(f, t, \Omega)_p \le C_2 h(t) \tag{3.9}$$

hold, then for all $\gamma \in M$, $\gamma > 1$, $1 \le p \le \infty$,

$$C_3 h\left(\gamma^{-1}\right) \le E(f, S_\gamma, \Omega)_p \le C_4 h\left(\gamma^{-1}\right). \tag{3.10}$$

PROOF. The upper bounds in (3.10) follow from Condition (A) and (3.9). Suppose now that f and $\{S_\gamma\}_{\gamma\in M}$ satisfy Conditions (A) and (B). Next, let $g_\gamma \in S_\gamma$ satisfy the inequality

$$\|f - g_\gamma\|_{L_p(\Omega)} \le 2E(f, S_\gamma, \Omega)_p. \tag{3.11}$$

Using (2.5), (2.6), and (3.11), we obtain

$$\begin{aligned} \omega_k(f, t, \Omega_1)_p &\le \omega_k(g_\gamma, t, \Omega_1)_p + \omega_k(f - g_\gamma, t, \Omega_1)_p \\ &\le \omega_k(g_\gamma, t, \Omega_1)_p + 2^{k+1} E(f, S_\gamma, \Omega)_p. \end{aligned} \tag{3.12}$$

Next, by (2.5), (2.6), (3.1), and (3.11), we have

$$\begin{aligned} \omega_k(g_\gamma, \gamma^{-1}, \Omega)_p &\le \omega_k\left(f, \gamma^{-1}, \Omega\right)_p + 2^{k+1} E(f, S_\gamma, \Omega)_p \\ &\le \omega_k\left(f, \gamma^{-1}, \Omega\right)_p + 2^{k+1}\omega_k\left(f, \gamma^{-1}, \Omega\right)_p \le C\omega_k\left(f, \gamma^{-1}, \Omega\right)_p. \end{aligned} \tag{3.13}$$

Then using (3.2), (3.10), (3.12), and (3.13), we obtain

$$\begin{aligned} E(f, S_\gamma, \Omega)_p &\ge 2^{-k-1}(\omega_k(f, t, \Omega_1)_p - \omega_k(g_\gamma, t, \Omega_1)_p) \\ &\ge C\left(C_1 h(t) - (\gamma t)^k \omega_k\left(g_\gamma, \gamma^{-1}, \Omega\right)_p\right) \ge C\left(C_1 h(t) - (\gamma t)^k h\left(\gamma^{-1}\right)\right). \end{aligned} \tag{3.14}$$

Next, by (3.8) and (3.14), we have for all $\gamma > 1$, $0 < \gamma t \le 1$,

$$E(f, S_\gamma, \Omega)_p \ge C\left(C_1(\gamma t)^\mu h\left(\gamma^{-1}\right) - (\gamma t)^k h\left(\gamma^{-1}\right)\right) = C(\gamma t)^\mu h\left(\gamma^{-1}\right)\left(C_1 - (\gamma t)^{k-\mu}\right), \tag{3.15}$$

where $0 < C_1 \le 2$. Putting $t = (C_1/2)^{1/k-\mu}\gamma^{-1}$, we obtain from (3.15)

$$E(f, S_\gamma, \Omega)_p \ge Ch\left(\gamma^{-1}\right),$$

giving the lower bounds in (3.10). ∎

REMARK 3. We can prove that (3.9) is equivalent to (3.10) if and only if $1 < H < 2^k$. The similar results for $p = \infty$, $\Omega = \Omega_1 = \mathbf{T}$ and subspaces of trigonometric polynomials have been obtained by Stechkin [20], Lozinskii [21], and Bari and Stechkin [22]. All these results can be generalized to normed spaces satisfying certain conditions (in the spirit of Butzer and Scherer [23]).

REMARK 4. In the proof of the theorem, use was made of a method due to [20].

Below we consider some applications of Theorem 2 to approximations by trigonometric and algebraic polynomials, entire functions of exponential type and splines.

(a) APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS. Let $\mathcal{T}_n$ be the class of all trigonometric polynomials with complex coefficients of degree at most n. Conditions (A) and (C) hold for $\Omega = \Omega_1 = \mathbf{T}$, $M = \mathbf{N}$, and $S_\gamma = \mathcal{T}_n$ (see [18, pp. 215, 273; 19, p. 205]). Next, the function $h(t) = h_k(t,\pi) = t^{(\lambda+1/p)/(1+\alpha)}$ satisfies the H-condition for $H = 2^{(\lambda+1/p)/(1+\alpha)} < 2^k$, $k > (\lambda+1/p)/(1+\alpha)$. Thus, Theorems 1 and 2 yield the following result.

COROLLARY 1. *For $\lambda > 0$, $\alpha > 0$, $p \in [1,\infty]$, $s \in \mathbf{N} \cup \{0\}$, $0 \le i \le 2$, $n \in \mathbf{N}$,*

$$C_1 n^{-(\lambda+1/p)/(1+\alpha)} \le E(\psi_{\lambda,\alpha,i,s}, \mathcal{T}_n, \mathbf{T})_p \le C_2 n^{-(\lambda+1/p)/(1+\alpha)}. \tag{3.16}$$

Let $c_n(f) = \pi^{-1}\int_{-\pi}^{\pi} f(\tau)e^{-in\tau}\,d\tau$, $n = 0, \pm1, \dots$ denote the Fourier coefficients of $f \in L_1(\mathbf{T})$.

COROLLARY 2.

(a) *For $\lambda > 0$, $\alpha > 0$, $p \in [1,\infty]$, $s \in \mathbf{N} \cup \{0\}$, $0 \le i \le 2$, $n \in \mathbf{N}$,*

$$|c_n(\psi_{\lambda,\alpha,i,s})| \le C n^{-(\lambda+1)/(1+\alpha)}. \tag{3.17}$$

(b) *If $\varphi^*_{\lambda,\alpha,i} = \varphi_{\lambda,\alpha,i}$ for $|x| < \pi$ is a 2π-periodic function, then for $0 < \lambda \le \alpha$, $0 \le i \le 2$, $n \in \mathbf{N}$,*

$$\left|c_n\left(\varphi^*_{\lambda,\alpha,i}\right)\right| \le C n^{-(\lambda+1)/(1+\alpha)}. \tag{3.18}$$

PROOF. Inequality (3.17) easily follows from (3.16) and the inequality

$$|c_n(f)| \le \pi^{-1} E(f, \mathcal{T}_{n-1}, L_1(\mathbf{T})). \tag{3.19}$$

Next, using (2.1) and (2.6), we obtain for $n \ge k \ge 2$

$$\omega_k\left(\varphi^*_{\lambda,\alpha,i}, n^{-1}, \mathbf{T}\right)_1 \le \omega_k\left(\varphi_{\lambda,\alpha,i}, n^{-1}, \left[-\pi + \frac{k}{n}, \pi - \frac{k}{n}\right]\right)_1 + \frac{C}{n} \le C n^{-(\lambda+1)/(1+\alpha)}. \tag{3.20}$$

Then, (3.18) follows from (3.19), (3.20), and the Jackson Theorem. ∎

REMARK 5. Inequality (3.18) for $\lambda = \alpha = 1$, $i = 2$ was obtained by Bray [26, p. 230].

(b) APPROXIMATION BY SPLINES. Let $S_{n,k}$ be the class of all 2π-periodic splines of order k and of defect 1 with respect to the uniform partition $\{l\pi/n\}_{l=1}^{2n}$. Condition (A) for $\Omega = \Omega_1 = \mathbf{T}$, $M = \mathbf{N}$ and $S_\gamma = S_{n,k}$ holds for all $p \in [1,\infty]$, [19, p. 356], and Condition (C) is valid for $p = 1, 2, \infty$ [24, pp. 220–236]. The following analogue of Corollary 1 is a consequence of Theorems 1 and 2.

COROLLARY 3. *For $\lambda > 0$, $\alpha > 0$, $p = 1, 2, \infty$, $k > (\lambda + 1/p)/(1+\alpha)$, $s \in \mathbf{N} \cup \{0\}$, $0 \le i \le 2$, $n \in \mathbf{N}$,*

$$C_1 n^{-(\lambda+1/p)/(1+\alpha)} \le E(\psi_{\lambda,\alpha,i,s}, S_{n,k}, \mathbf{T})_p \le C_2 n^{-(\lambda+1/p)/(1+\alpha)}.$$

(c) APPROXIMATION BY ALGEBRAIC POLYNOMIALS. Let $\mathcal{P}_n$ be the class of all algebraic polynomials with complex coefficients of degree at most n. Conditions (A) and (C) hold for $k = 1$, $p = \infty$, $\Omega = [-a, a]$, $\Omega_1 = [-a_1, a_1]$, $0 < a_1 < a$, $M = \mathbf{N}$, $S_\gamma = \mathcal{P}_n$. In this case, Condition (A) is the classical Jackson Theorem [18, p. 254], and Condition (C) easily follows from its periodic analogue [24, p. 148]. If $k - 1 + 1/p > 0$, then inequalities (3.2) and (3.3) for algebraic polynomials are unknown. That is why the proof of the following result is based on Corollary 2.

COROLLARY 4. *For* $\lambda > 0$, $\alpha > 0$, $p \in [1, \infty]$, $0 \le i \le 2$, $n \in \mathbf{N}$,

$$C_1 n^{-(\lambda+1/p)/(1+\alpha)} \le E(\varphi_{\lambda,\alpha,i}, \mathcal{P}_n, [-a,a])_p \le C_2 n^{-(\lambda+1/p)/(1+\alpha)}. \tag{3.21}$$

PROOF. It is sufficient to prove (3.21) for $a = 1$. The following formula is easy to verify from the definition of $E(f, \mathcal{P}_n, [-a,a])_p$:

$$E(\varphi_{\lambda,\alpha,i}, \mathcal{P}_n, [-1,1])_p = \begin{cases} 2^{-1/p} \inf_{T \in \mathcal{T}_n} \left(\int_{-\pi}^{\pi} |\psi_{\lambda,\alpha,i,0}(\tau) - T(\tau)|^p |\cos\tau| \, d\tau \right)^{1/p}, & 1 \le p < \infty, \\ E(\psi_{\lambda,\alpha,i,0}, \mathcal{T}_n, \mathbf{T})_\infty, & p = \infty. \end{cases} \tag{3.22}$$

Next, (3.22) shows that for $1 \le p \le \infty$,

$$E(\varphi_{\lambda,\alpha,i}, \mathcal{P}_n, [-1,1])_p \le 2^{-1/p} E(\psi_{\lambda,\alpha,i,0}, \mathcal{T}_n, \mathbf{T})_p, \tag{3.23}$$

$$\begin{aligned} E(\varphi_{\lambda,\alpha,i}, \mathcal{P}_n, [-1,1])_p &\ge 2^{-1/p} \inf_{T \in \mathcal{T}_n} \left(\int_{-\pi}^{\pi} |\psi_{\lambda,\alpha,i,1}(\tau) - \cos\tau T(\tau)|^p \, d\tau \right)^{1/p} \\ &\ge 2^{-1/p} E(\psi_{\lambda,\alpha,i,1}, \mathcal{T}_{n+1}, \mathbf{T})_p. \end{aligned} \tag{3.24}$$

Then (3.16), (3.23), and (3.24) yield (3.21). ∎

(d) APPROXIMATION BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE. Let B_σ be the class of all entire functions of exponential type $\sigma > 0$. Condition (A) holds for $\Omega = \Omega_1 = \mathbf{R}$, $M = \{x \in \mathbf{R} : x > 0\}$, $S_\gamma = B_\sigma$, and measurable functions f, satisfying the inequality $|f(x)| \le C(1 + |x|^N)$, $N \ge 0$, $x \in \mathbf{R}$ (see [18, p. 259; 25]). Inequality (3.3) for $g \in B_\sigma \cap L_p(\mathbf{R})$ was obtained by [18, p. 217]. The following lemma shows that Condition (C) is valid for all $g \in B_\sigma$ with $\omega_k(g, t, \mathbf{R})_p < \infty$.

LEMMA 2. *If* $g \in B_\sigma$ *and for all* $h \in (0, \pi/(32k\sigma))$, $\|\Delta_h^k g\|_{L_p(\mathbf{R})} < \infty$, *then*

$$\left\| g^{(k)} \right\|_{L_p(\mathbf{R})} \le 2h^{-k} \left\| \Delta_h^k g \right\|_{L_p(\mathbf{R})}, \qquad 1 \le p \le \infty. \tag{3.25}$$

PROOF. Let $g \in B_\sigma$ and $h \in (0, \pi/(32k\sigma))$. First, prove (3.25) for $p = \infty$. It is easy to see that there exists an increasing sequence $x_s \in (s\pi/8\sigma, s\pi/8\sigma + kh)$, $s = 0, \pm1, \pm2, \dots$, satisfying the relations

$$\sup_s \left| g^{(k)}(x_s) \right| \le h^{-k} \sup_s \left| \Delta_h^k g\left(\frac{s\pi}{8\sigma}\right) \right| \le h^{-k} \left\| \Delta_h^k g \right\|_{L_\infty(\mathbf{R})}, \tag{3.26}$$

$$\sup_s \left(x_s - \frac{s\pi}{8\sigma} \right) \le \frac{\pi}{32\sigma}, \qquad \inf_{s \ne m} |x_s - x_m| \ge \frac{3\pi}{32\sigma}, \tag{3.27}$$

$$\sup_s (x_{s+1} - x_s) \le \frac{5\pi}{32\sigma} < \frac{1}{2\sigma}. \tag{3.28}$$

Now, using (3.26), (3.27), and a generalization of Cartwright's theorem obtained by Duffin and Schaeffer [27], we have $\|g^{(k)}\|_{L_\infty(\mathbf{R})} < \infty$. Then, by the Bernstein inequality for entire functions of exponential type [18, p. 208],

$$\left\| g^{(k)} \right\|_{L_\infty(\mathbf{R})} \le \sup_s \left| g^{(k)}(x_s) \right| + \sigma \left\| g^{(k)} \right\|_{L_\infty(\mathbf{R})} \sup_s (x_{s+1} - x_s). \tag{3.29}$$

Finally, (3.26), (3.28), and (3.29) yield

$$\left\| g^{(k)} \right\|_{L_\infty(\mathbf{R})} \le 2h^{-k} \left\| \Delta_h^k g \right\|_{L_\infty(\mathbf{R})}. \tag{3.30}$$

To prove (3.25) for $1 \le p < \infty$, we use a method due to Stein [24,28]. First note that, by the Nikolskii inequality [18, p. 233], $\Delta_h^k g(x) \in L_\infty(\mathbf{R})$. Then, from (3.30) we have $g^{(k)} \in L_\infty(\mathbf{R})$. Therefore, g satisfies the inequality

$$|g(x)| \le C(1+|x|)^k, \qquad x \in \mathbf{R}. \tag{3.31}$$

Further, by (3.31), the convolution $(\varphi * g)(x) = \int_{-\infty}^{\infty} \varphi(t) g(x-t)\,dt$ belongs to B_σ for every $\varphi \in L_{p'}(\mathbf{R})$, $p' = p/(p-1)$, $1 \le p < \infty$, with a bounded support (cf. [18, p. 259].) Therefore, the function $\Delta_h^k(\varphi * g)(x) = (\varphi * \Delta_h^k g)(x)$ belongs to $B_\sigma \cap L_\infty(\mathbf{R})$, and using (3.30) we obtain

$$\begin{aligned} \left\| g^{(k)} \right\|_{L_p(\mathbf{R})} &= \sup_{x\in\mathbf{R}} \sup_{\|\varphi\|_{L_{p'}(\mathbf{R})}\le 1} \left|\varphi * g^{(k)}(x)\right| = \sup_{\|\varphi\|_{L_{p'}(\mathbf{R})}\le 1} \sup_{x\in\mathbf{R}} \left|\varphi * g^{(k)}(x)\right| \\ &\le 2h^{-k} \sup_{\|\varphi\|_{L_{p'}(\mathbf{R})}\le 1} \sup_{x\in\mathbf{R}} \left|\varphi * \Delta_h^k g(x)\right| \le 2h^{-k} \left\|\Delta_h^k g\right\|_{L_p(\mathbf{R})}. \end{aligned} \tag{3.32}$$

Relations (3.30) and (3.32) yield (3.25). ∎

The function $h(t) = h_{k,i}(t,\infty) = t^{(\lambda+1/p)/(1+\alpha)}$ satisfies the H-condition for $k > \lambda + 1/p$, $1 < H < 2^k$. Thus, Theorems 1 and 2 yield the following result.

COROLLARY 5. *For $\lambda > 0$, $\alpha > 0$, $p \in [1,\infty]$, $0 \le i \le 2$, $\sigma > 0$,*

$$C_1\sigma^{-(\lambda+1/p)/(1+\alpha)} \le E(\varphi_{\lambda,\alpha,i}, B_\sigma, \mathbf{R})_p \le C_2\sigma^{-(\lambda+1/p)/(1+\alpha)}.$$

REFERENCES

1. Ch.J. de la Vallée-Poussin, Sur les polynômes d'approximation et la representation approchee d'un angle, *Bull. Acad. Sc. Belgique* **808** (1910).
2. S.N. Bernstein, Sur la meilleure approximation de $|x|$ par des polynômes des degrés donnés, *Acta Math.* **37**, 1–57 (1913).
3. S.N. Bernstein, On the best approximation of $|x|^p$ by means of polynomials of extremely high degree, *Izv. Akad. Nauk SSSR,* (in Russian) **2**, 169–180 (1938).
4. R.S. Varga and A.J. Carpenter, On the Bernstein conjecture in approximation theory, *Const. Approx.* **1**, 333–348 (1985).
5. D.J. Newman, Rational approximation to $|x|$, *Michigan Math. J.* **11**, 11–14 (1964).
6. N.S. Vjacheslavov, On the uniform approximation of $|x|$ by rational functions, *Soviet Math. Dokl.,* (in Russian) **16**, 100–104 (1975).
7. H. Stahl, Best uniform approximation of x^α on $[0,1]$, *Bull. Amer. Math. Soc.* **28**, 116–122 (1993).
8. R.A. DeVore and K. Scherer, Variable knots, variable degree spline approximation to x^β, In *Quantitative Approximation,* (Edited by R.A. DeVore and K. Scherer), pp. 121–131, Academic Press, New York, (1980).
9. E.S. Belinskii, Decomposition theorems and approximations by a "floating" system of exponentials, *Trans. Amer. Math. Soc.* (to appear).
10. S.N. Bernstein, On the best approximation of $|x-c|^p$, *Dokl. Akad. Nauk SSSR,* (in Russian) **18**, 379–384 (1938).
11. S.N. Bernstein, A new result and some formulae for best approximation, *Dokl. Akad. Nauk SSSR,* (in Russian) **54**, 667–668 (1946).
12. S.N. Bernstein, On the best approximation of the function $\int_0^\infty |y|^s\,d\psi(s)$ on the segment $[-1,1]$, *Izv. Akad. Nauk SSSR* (in Russian) **10**, 185–196 (1946).
13. S.M. Nikolskii, On the best approximation of functions, the s^{th} derivative of which has discontinuities of the first kind, *Dokl. Akad. Nauk SSSR,* (in Russian) **55**, 99–102 (1947).
14. S.M. Nikolskii, On the best mean approximation by polynomials of the function $|a-x|^s$, *Izv. Akad. Nauk SSSR,* (in Russian) **11**, 139–180 (1947).
15. Yu.A. Brudnyi, Adaptive approximation of functions with singularities, *Trans. Moscow Math. Soc.*, 123–186 (1994).
16. P.P. Petrushev and V.A. Popov, *Rational Approximation of Real Functions,* Cambridge University Press, Cambridge, (1987).
17. M.I. Ganzburg, Best approximation of functions like $|x|^\lambda \exp(-A|x|^{-\alpha})$, *J. Approx. Theory* **92**, 379–410 (1998).
18. A.F. Timan, *Theory of Approximation of Functions of a Real Variable,* Pergamon Press, New York, (1983).
19. R.A. DeVore and G.G. Lorentz, *Constructive Approximation,* Springer-Verlag, Berlin, (1993).

20. S.B. Stechkin, On the order of the best approximations of continuous functions, *Izv. Akad. Nauk SSSR,* (in Russian) **15**, 219–242 (1951).
21. S.M. Lozinskii, The converse of Jackson's theorems, *Dokl. Akad. Nauk SSSR,* (in Russian) **83**, 645–647 (1952).
22. N.K. Bari and S.B. Stechkin, The best approximation and differential properties of two conjugate functions, *Trudy Mosk. Matem. Ob-va* (in Russian) **5**, 483–522 (1956).
23. P.L. Butzer and K. Scherer, On the fundamental approximation theorems of D. Jackson, S.N. Bernstein and theorems of M. Zamansky and S.B. Stechkin, *Aequationes Math.* **3**, 170–185 (1969).
24. N.P. Korneichuk, V.F. Babenko and A.A. Ligun, *Extremal Properties of Polynomials and Splines,* (in Russian), Naukova Dumka, Kiev, (1992).
25. M.I. Ganzburg, Multidimensional Jackson theorems, *Siberian Math. J.* **22** (2), 223–231 (1981).
26. A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, (1988).
27. R. Duffin and A.C. Schaeffer, Power series with bounded coefficients, *Amer. J. Math.* **67**, 141–154 (1945).
28. E. Stein, Functions of exponential type, *Ann. Math.* **65**, 582–592 (1957).